$\mathcal{H}$-Matrix Parallelisation

Ronald Kriemann

Max Planck Institute for Mathematics

in the

Sciences Leipzig

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Overview

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Introduction

Problem

- Matrix construction is very expensive for BEM problems,
- solving equation system often needs preconditioning, but $\mathcal{H}$-LU factorisation is also expensive.

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Exploit parallel capabilities of PCs or workstations to accelerate $\mathcal{H}$-arithmetics.

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- solving equation system often needs preconditioning, but $\mathcal{H}$-LU factorisation is also expensive.

Goal

Exploit parallel capabilities of PCs or workstations to accelerate $\mathcal{H}$-arithmetics.

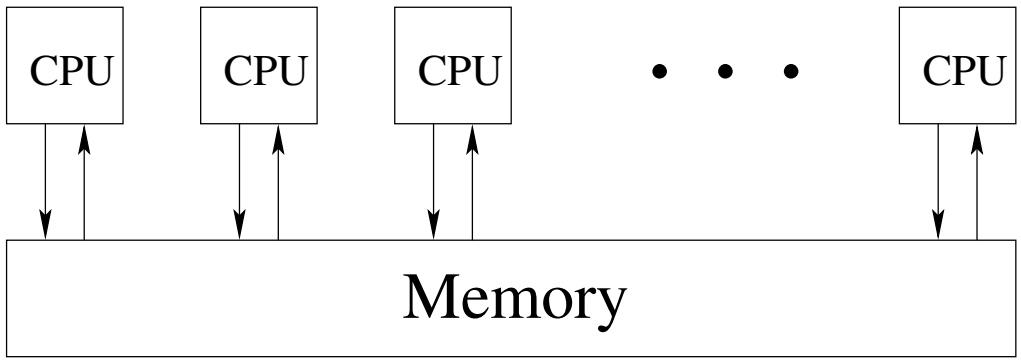
Conditions

- parallelise only for few processors,
- recycle existing algorithm,
- achieve high parallel speedup

Overview

System Architecture

Workstations and small compute servers are usually systems with a shared memory, e.g. all p processors have direct access to the same memory:



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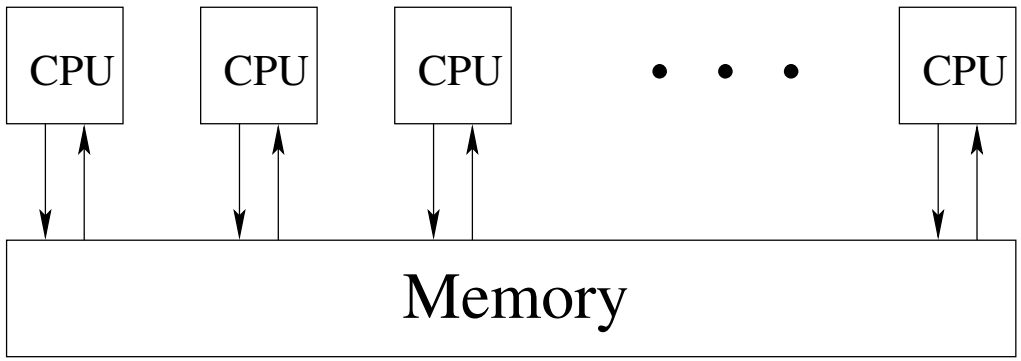

Consequences

- simplified programming because no communications involved,
- but protection of critical resources necessary, e.g. simultaneous access to the same memory position

Technical Prerequisites

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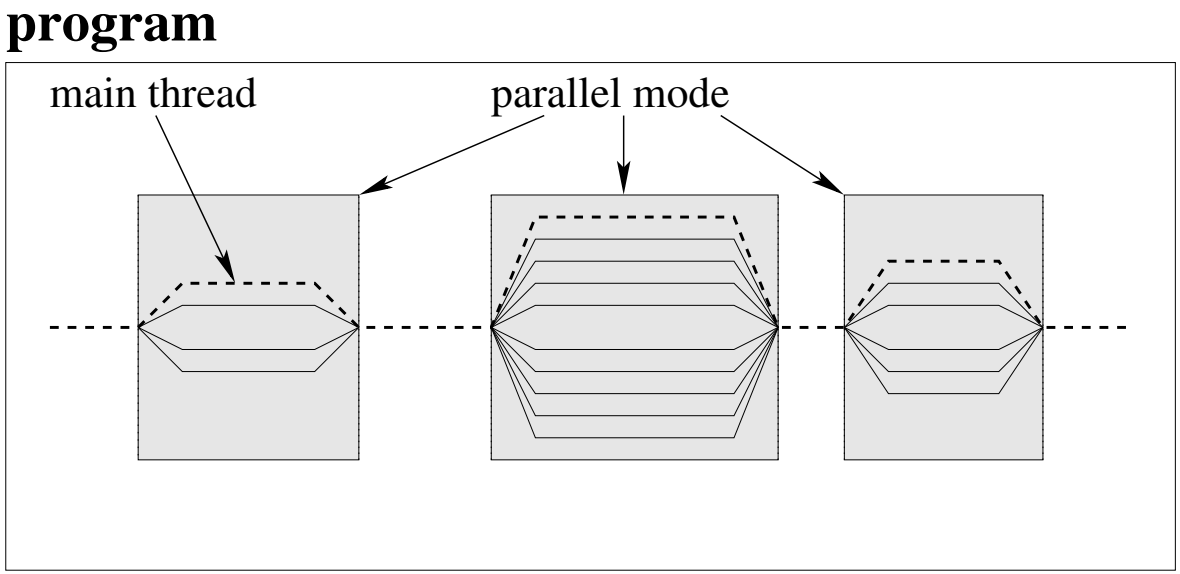

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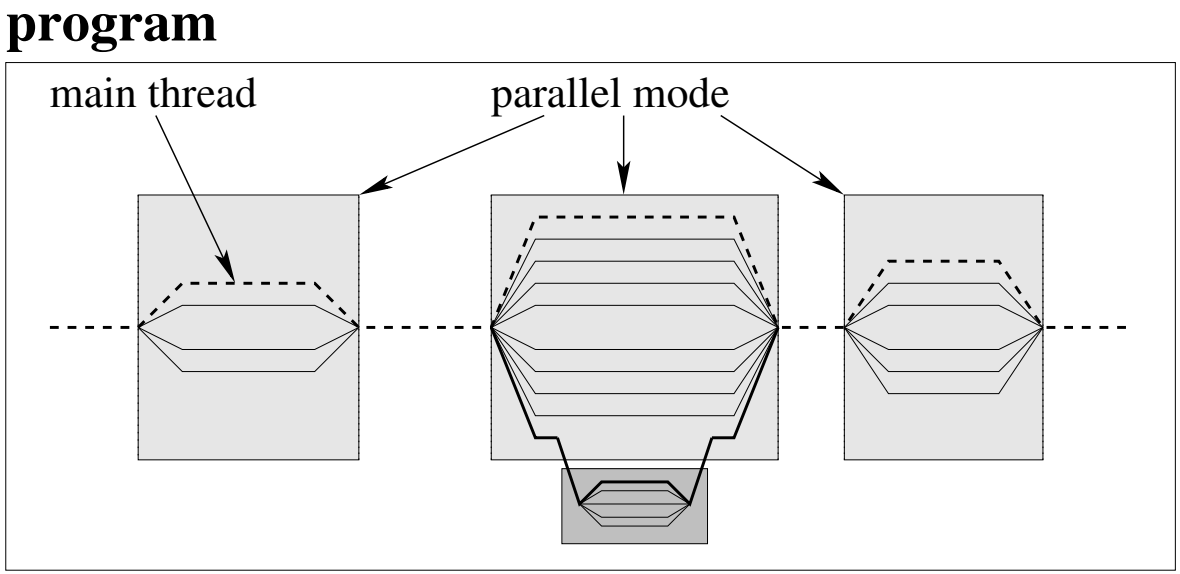

Also possible is nested parallelism: starting new threads from other threads.

Technical Prerequisites

Mutices

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Example: compute $A := A + \sum_{i=1}^{4} A_i$, with matrices A and $A_i, i = 1, \ldots, 4$

On thread 1:

$$T_1 := A_1 + A_2;$$

$$A := A + T_1;$$

On thread 2:

$$T_2 := A_3 + A_4;$$

$$A := A + T_2;$$

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On thread 1:

$T_1 := A_1 + A_2;$
lock mutex m;
$A := A + T_1;$
unlock mutex m;

On thread 2:

$T_2 := A_3 + A_4;$
lock mutex m;
$A := A + T_2;$
unlock mutex m;

with shared mutex m.

Thread Implementations

How to access threads in a program, e.g. how to start new threads and synchronise them via mutices?

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Widely used implementations:

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- POSIX threads:
  - \+ powerful, almost everywhere available
  - − complicate interface
- OpenMP:
  - \+ simple interface
  - − mainly developed for loop parallelisation, needs compiler support

Technical Prerequisites

OpenMP

Language enhancements for C/C++ and FORTRAN for thread creation, loop parallelisation and synchronisation.

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Example for linear combination of vectors: $y := y + \alpha x$:

```
#pragma omp parallel for
    for ( i = 0; i < n; i++ )
        y[i] := y[i] + alpha * x[i];
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At the **pragma** directive p threads are started. The loop is automatically parallelised and after finishing, all threads are synchronised. If no OpenMP support is available, the directive is ignored by the compiler.

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OpenMP is supported by all major compilers: GNU, Intel, Sun, Microsoft but with varying degree, e.g. nested parallelism.

Other OpenMP Directives

#pragma omp parallel

Starts p threads executing the following code block.

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OpenMP Functions and Types

For explicit mutices, the type `omp_lock_t` is defined by OpenMP with the functions

```
// lock given mutex
void  omp_set_lock    ( omp_lock_t * mutex );
// unlock given mutex
void  omp_unset_lock  ( omp_lock_t * mutex );
```

Overview

Matrix Construction

Let $\mathcal{I}$ be an index set with $\#\mathcal{I} = n$, $T(\mathcal{I})$ a cluster tree over $\mathcal{I}$ and $T(\mathcal{I} \times \mathcal{I})$ a block cluster tree over $\mathcal{I} \times \mathcal{I}$.

Sequential Algorithm without Hierarchy

Build matrix blocks only for leaves in block cluster tree:

for $(t, s) \in \mathcal{L}(T(\mathcal{I} \times \mathcal{I}))$ **do**
    if (t, s) is admissible **then** build low rank matrix;
    else build dense matrix;
endfor;

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Properties

- all matrix blocks can be built independently,
- $\#\mathcal{L}(T(\mathcal{I} \times \mathcal{I})) \gg p$, e.g. enough to do for each processor and no explicit scheduling necessary.

Matrix Construction

Parallel Algorithm without Hierarchy

Apply OpenMP parallelisation directly to loop:

```
leaves = ... // list of leaves
#pragma omp parallel {
    while ( ! is_empty( leaves ) ) {
        // guard change of leaves set by mutex
        #pragma omp critical {
            cluster = head( leaves );
            leaves  = tail( leaves );
        }

        if ( is_adm( cluster ) )
            M = build_lowrank( cluster );
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            M = build_dense( cluster );
}   }
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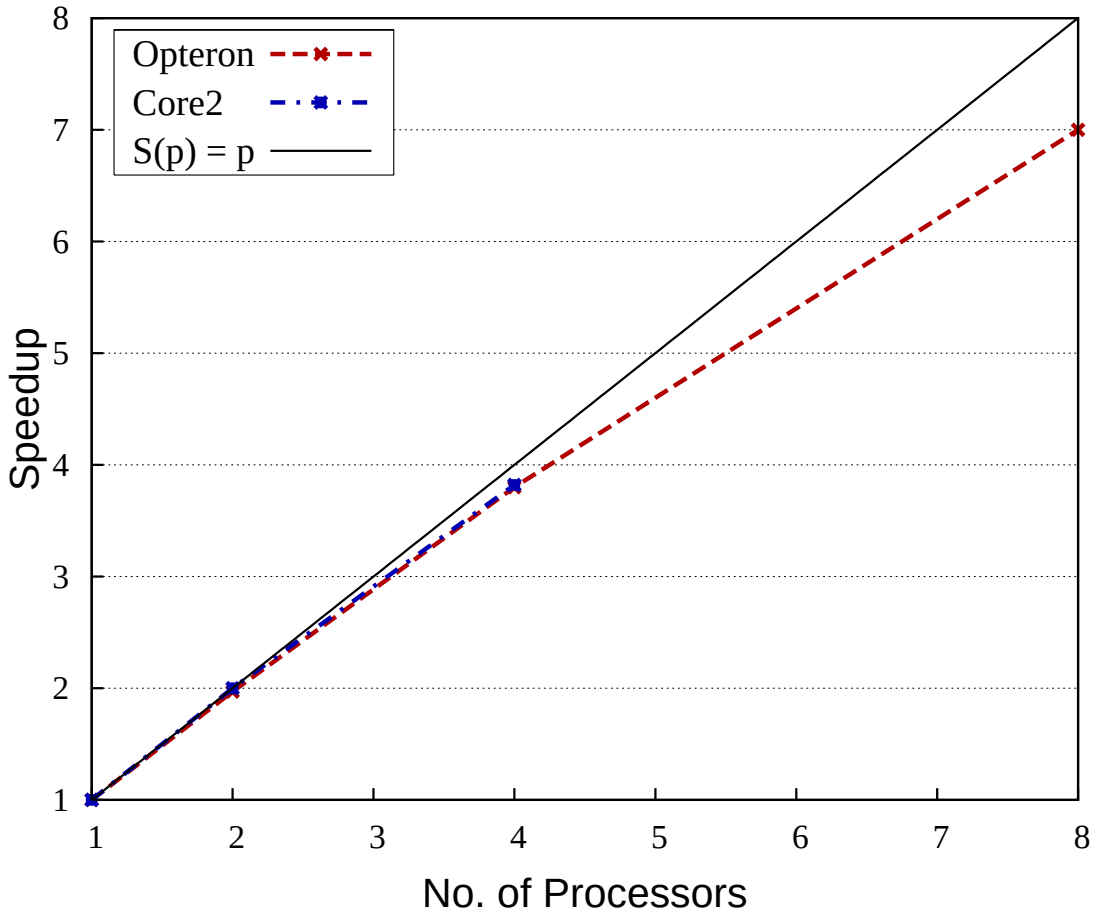
Building block matrices is cheap and can be done sequentially.

Numerical Experiments (Helmholtz DLP)



Overview

Preconditioning

We are looking for a good preconditioner for the linear equation system

$$Ax = y$$

where A is an $\mathcal{H}$-Matrix.

Good guess for a preconditioner: $P = A^{-1}$. Therefore use $\mathcal{H}$-matrix inverse $\tilde{P} = A^{\sim 1}$.

For iterative schemes one only needs matrix vector multiplication, therefore $\mathcal{H}$-LU factorisation is sufficient for evaluating $A^{\sim 1}$ and cheaper to compute.

Preconditioning

Assume A has 2×2 block structure:

$$\begin{pmatrix} L_{11} & \\ L_{21} & L_{22} \end{pmatrix} \cdot \begin{pmatrix} U_{11} & U_{12} \\ & U_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

Sequential Algorithm

Solving the above system, one obtains:

$$\begin{aligned} L_{11}U_{11} &= A_{11}, & L_{11}U_{12} &= A_{12}, \\ L_{21}U_{11} &= A_{21}, & L_{22}U_{22} &= A'_{22} \end{aligned}$$

with

$$A'_{22} = A_{22} - L_{21}U_{12}.$$

This involves two recursive calls, two matrix solves and one multiplication (with addition).

Matrix Solve

Again, assume 2×2 block structure and consider $BU = C$ with known C and upper triangular U, e.g. $L_{21}U_{11} = A_{21}$ from above.

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \cdot \begin{pmatrix} U_{11} & U_{12} \\ & U_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

This leads to

$$\begin{aligned} B_{11}U_{11} &= C_{11}, & B_{12}U_{22} &= C'_{12}, \\ B_{21}U_{11} &= C_{21}, & B_{22}U_{22} &= C'_{22} \end{aligned}$$

with

$$C'_{12} = C_{12} - B_{11}U_{12} \quad \text{and} \quad C'_{22} = C_{22} - B_{21}U_{12}$$

involving four recursions and two multiplications.

Preconditioning

$\mathcal{H}$-LU factorisation and matrix solve only involves recursive calls and multiplications. Therefore, parallelising the multiplications, parallelises the $\mathcal{H}$-LU factorisation.

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Parallel Matrix Multiplication

$$C := \alpha AB + \beta C$$

Sequential algorithm for a $m \times m$ block matrix:

```
void mul ( alpha, A, B, beta, C ) {
   if ( is_blocked( A ) && is_blocked( B ) &&
        is_blocked( C ) )
      for ( i = 0; i < m; i++ )
         for ( j = 0; j < m; j++ )
            for ( l = 0; l < m; l++ )
               mul( alpha, A_il, B_lj, beta, C_ij );
   else
      C := alpha * A * B + beta * C;
}
```

Preconditioning

Parallel Matrix Multiplication

Collect all atomic multiplications into list and apply OpenMP parallelisation to list:

```
void mul_sim ( A, B, C, list ) {
   if ( is_blocked( A ) && is_blocked( B ) && is_blocked( C ) )
      for ( i, j, l = 0;   i, j, l < m;   i++, j++, l++ )
          mul_sim( A_il, B_lj, C_ij, list );
   else
      append( list, { C, A, B } ); }
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void mul ( alpha, A, B, beta, C ) {
    mul_sim( A, B, C, list );
    #pragma omp parallel {
        while ( ! is_empty( list ) ) {
            #pragma omp critical {
                product = head( list );
                list    = tail( list );
            }

            T = alpha * product.A * product.B;

            omp_set_lock( mutex( product.C ) ); // guard access to C
            product.C = beta * product.C + T;
            omp_unset_lock( mutex( product.C ) );
}   }   }
```

Preconditioning

Improved Parallel Matrix Multiplication

Collect products per matrix to reduce collisions:

```
void mul_sim2 ( A, B, C ) {
   if ( is_blocked( A ) && is_blocked( B ) && is_blocked( C ) )
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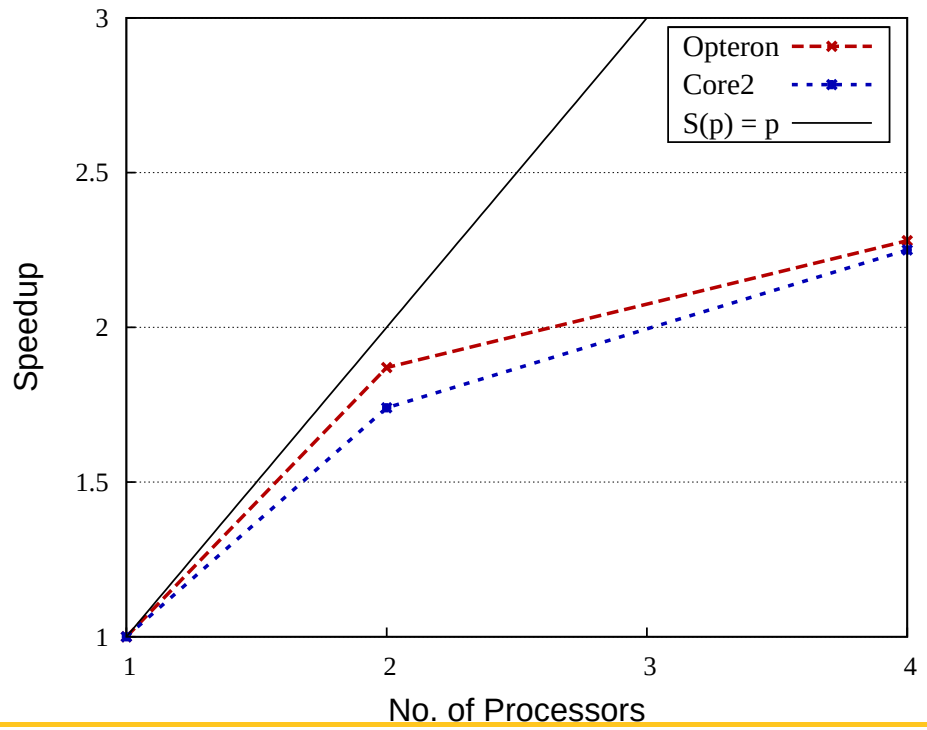

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Alternative preconditioning techniques:

- restrict to block diagonal format:
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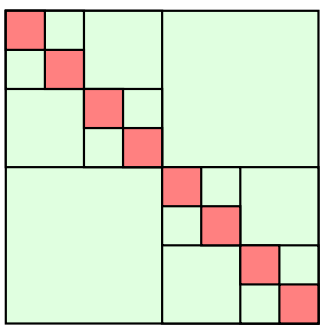
Successively remove off-diagonal blocks from $\mathcal{H}$-matrix with increasing number of processors:



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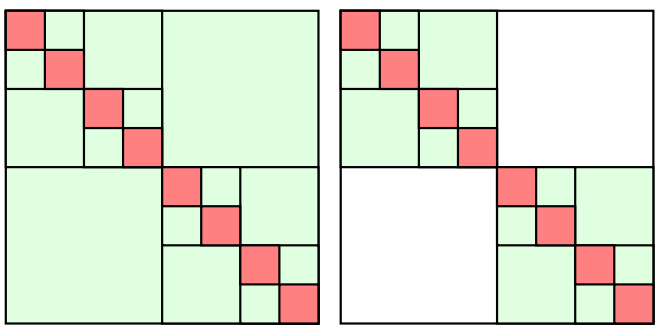

$p = 1$ $p = 2$

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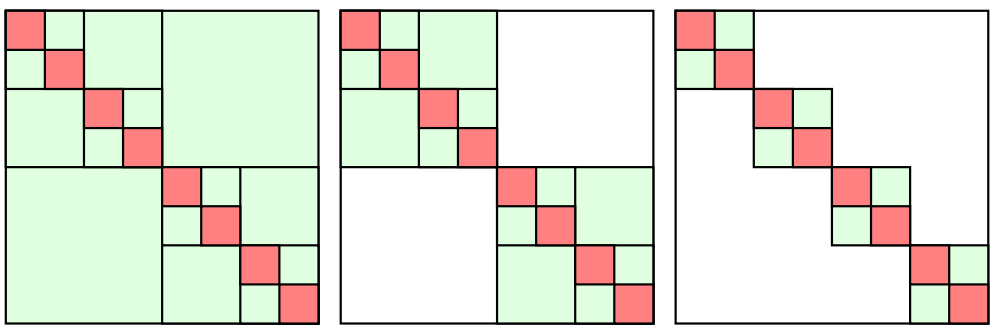

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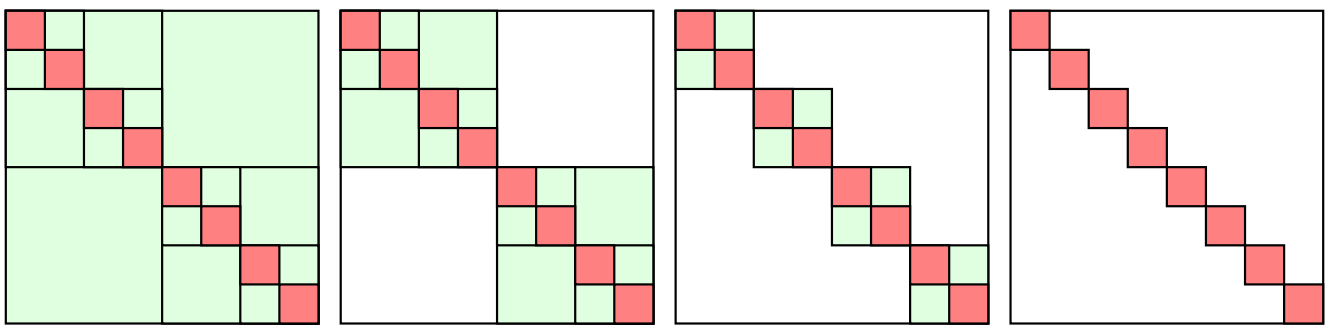


$p = 1$ $p = 2$ $p = 4$ $p = 8$

Preconditioning

Block Diagonal Preconditioner

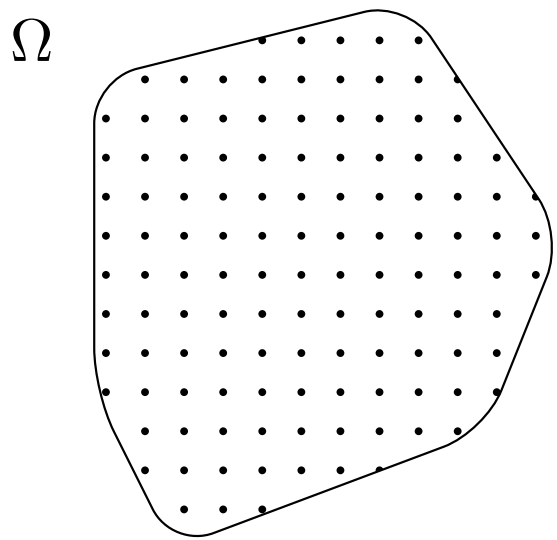
Assume 2×2 block structure of all block matrices in given $\mathcal{H}$-matrix. Then, the algorithm for the parallel LU factorisation for block diagonal matrices is:

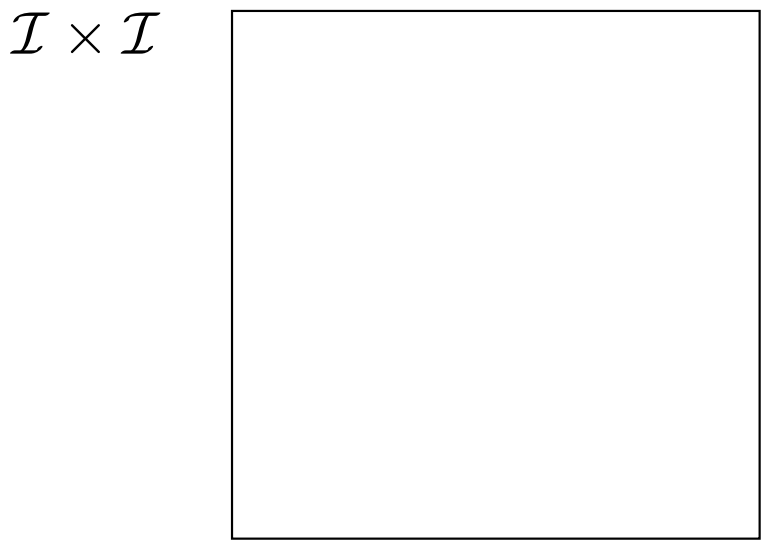
```
void blockdiag_LU ( p, A ) {
    if ( p == 1 ) LU( A );
    else {
        #pragma omp parallel for num_threads(2)
            for ( i = 0; i < 2; i++ )
                blockdiag_LU( p/2, A_ii );
}   }   }
```

The OpenMP option `num_threads(2)` ensures, that only two threads are started.
This approach requires support for nested parallelism in OpenMP implementation.

Nested Dissection

Assume a finite element discretisation for a partial differential equation, e.g. with piecewise linear ansatz functions. Since the support of the basis functions is local, one can find a subset Γ of $\mathcal{I}$, such that the remaining indices are decomposed into decoupled sets:





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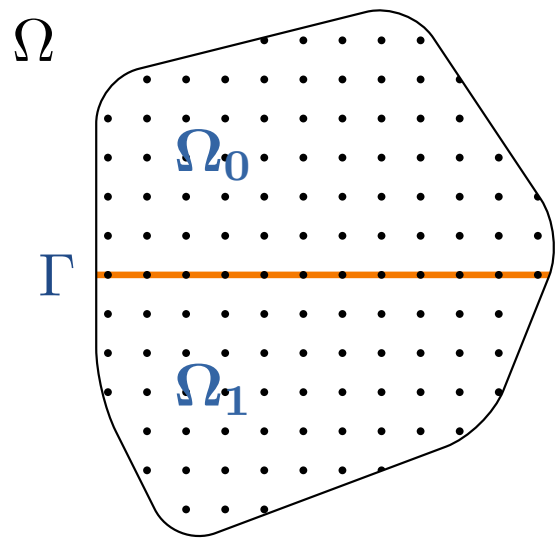

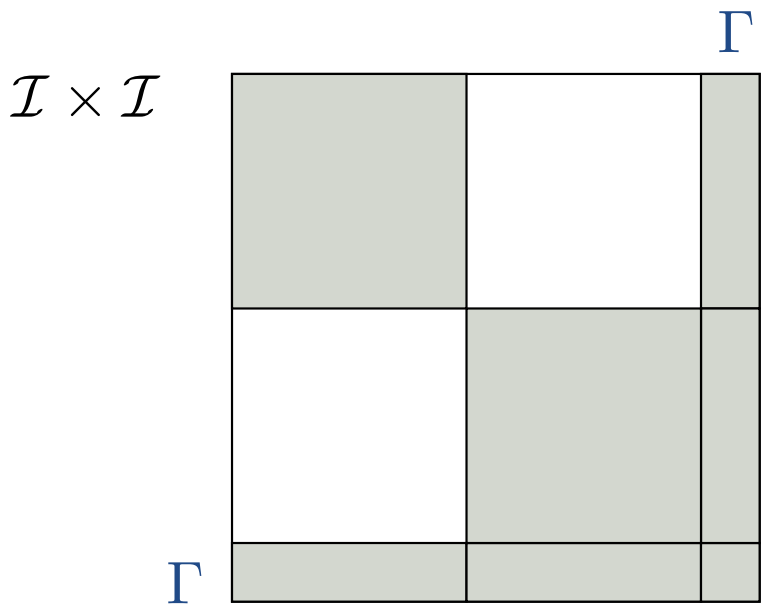

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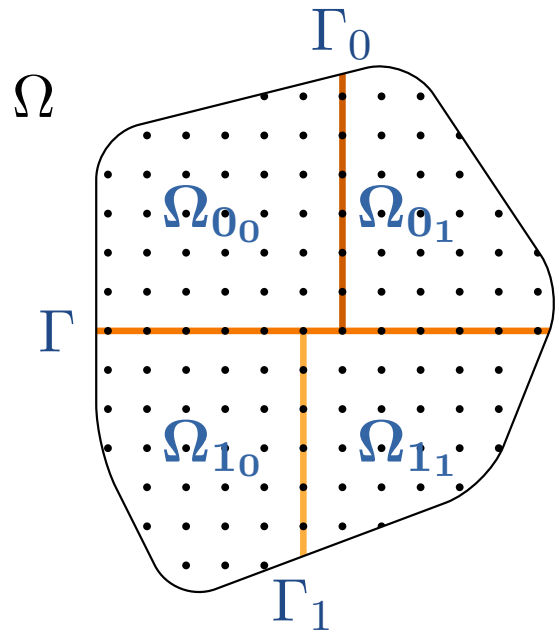

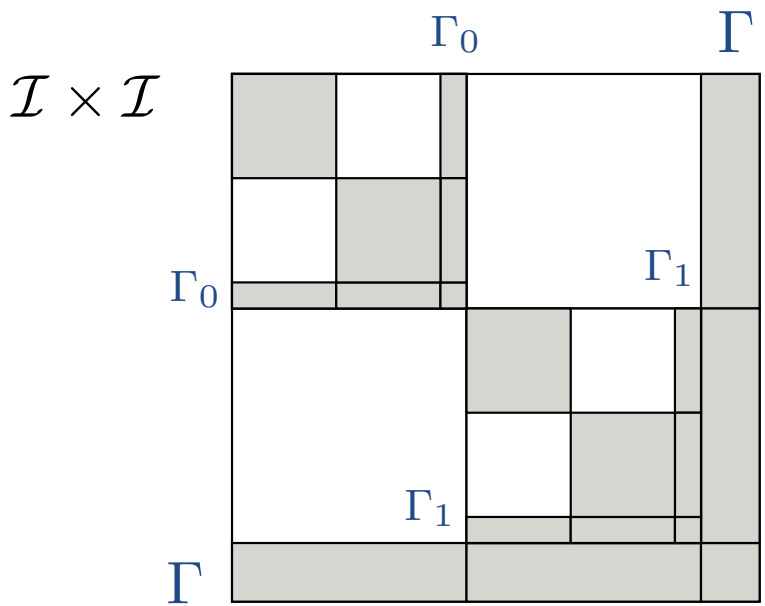

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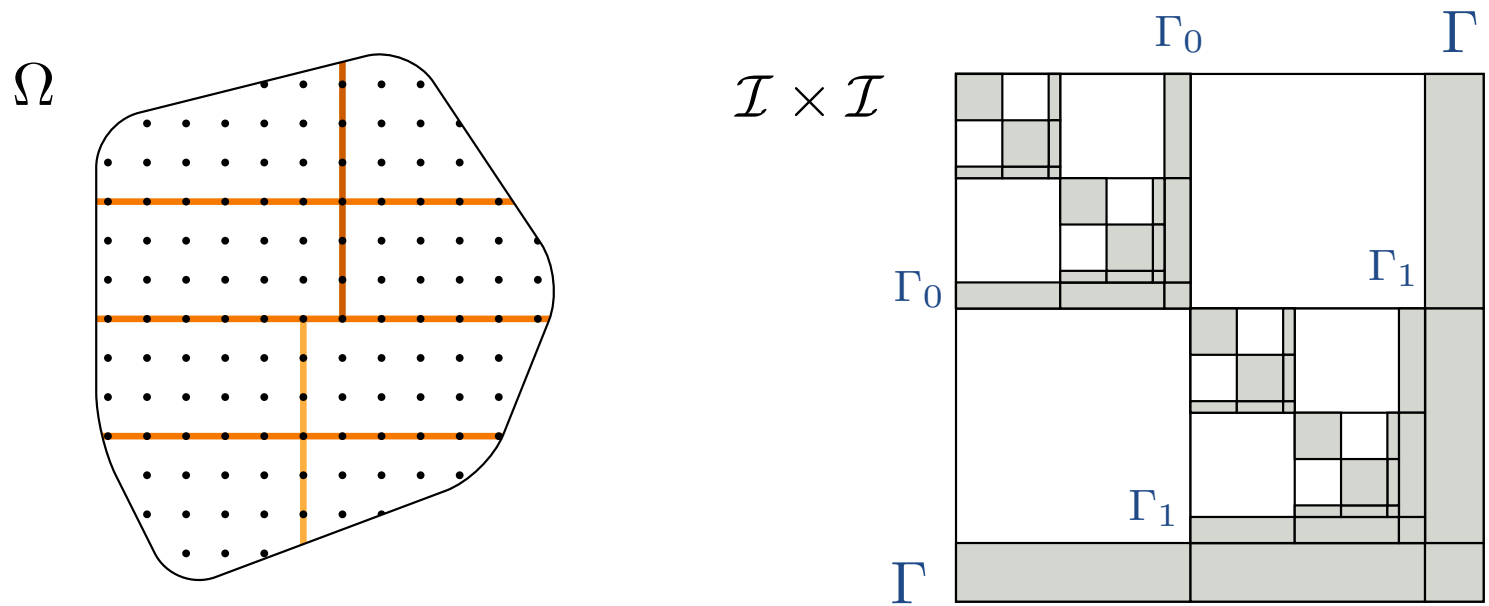

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Nested Dissection: LU factorisation

The L and U factors in a LU factorisation of A have the same structure as A:

$$\begin{pmatrix} L_{11} & & \\ & L_{22} & \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \cdot \begin{pmatrix} U_{11} & & U_{13} \\ & U_{22} & U_{23} \\ & & U_{33} \end{pmatrix} = \begin{pmatrix} A_{11} & & A_{13} \\ & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

leading to

$$\begin{aligned} L_{11}U_{11} &= A_{11} & L_{22}U_{22} &= A_{22} \\ L_{31}U_{11} &= A_{31} & L_{32}U_{22} &= A_{32} \\ L_{11}U_{13} &= A_{13} & L_{22}U_{23} &= A_{23} \end{aligned}$$

which can be handled independently and

$$L_{33}U_{33} = A'_{33} \quad \text{with} \quad A'_{33} = A_{33} - L_{31}U_{13} - L_{32}U_{23}.$$

Nested Dissection: LU factorisation

The L and U factors in a LU factorisation of A have the same structure as A:

$$\begin{pmatrix} L_{11} & & \\ & L_{22} & \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \cdot \begin{pmatrix} U_{11} & & U_{13} \\ & U_{22} & U_{23} \\ & & U_{33} \end{pmatrix} = \begin{pmatrix} A_{11} & & A_{13} \\ & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

leading to

$$\begin{aligned} L_{11}U_{11} &= A_{11} & L_{22}U_{22} &= A_{22} \\ L_{31}U_{11} &= A_{31} & L_{32}U_{22} &= A_{32} \\ L_{11}U_{13} &= A_{13} & L_{22}U_{23} &= A_{23} \end{aligned}$$

which can be handled independently and

$$L_{33}U_{33} = A'_{33} \quad \text{with} \quad A'_{33} = A_{33} - L_{31}U_{13} - L_{32}U_{23}.$$

$\#\Gamma$ should be small to have small sequential part.

Preconditioning

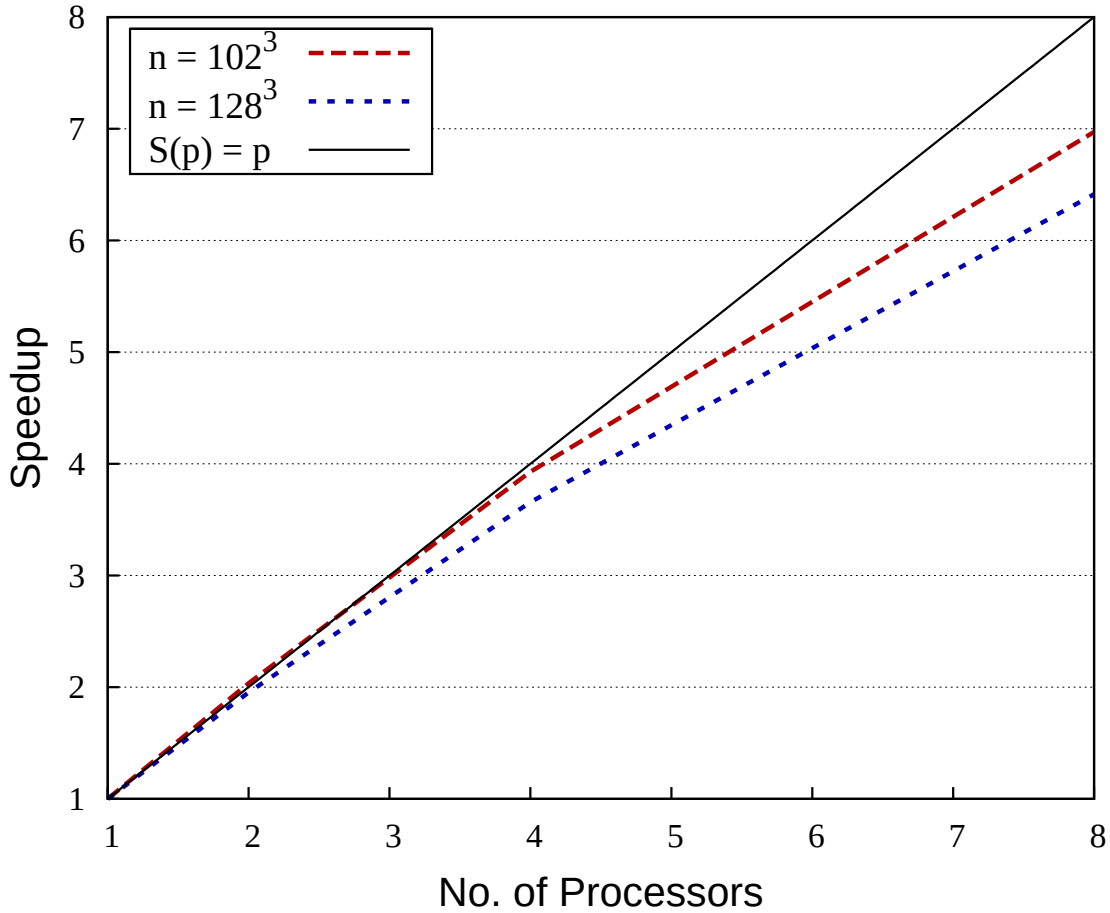
Nested Dissection: LU factorisation

```
void nd_LU ( p, A ) {
    if ( p == 1 ) LU( A );
    else {
        #pragma omp parallel for num_threads(2)
            for ( i = 0; i < 2; i++ ) {
                nd_LU( p/2, A_ii );
                nd_solve_lower( p/2, A_3i, A_ii );
                nd_solve_upper( p/2, A_ii, A_i3 );
                T_i = nd_mul( p/2, A_3i, A_i3 );

                omp_set_lock( mutex( A_33 ) );
                A_33 = A_33 - T_i;
                omp_unset_lock( mutex( A_33 ) );
        }
        LU( A_33 );
}   }
```

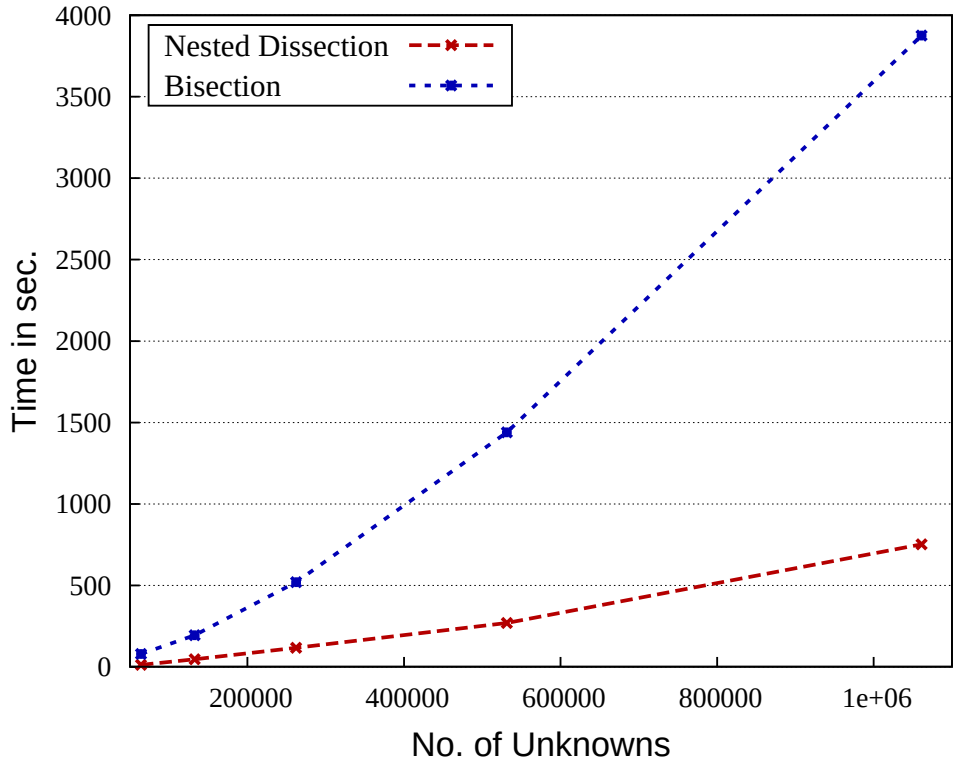
Nested Dissection: Numerical Experiments

Poisson problem in $\Omega = [0, 1]^3$



Nested Dissection: Remarks

- due to sparsity structure, nested dissection approach much faster than standard bisection even sequentially:



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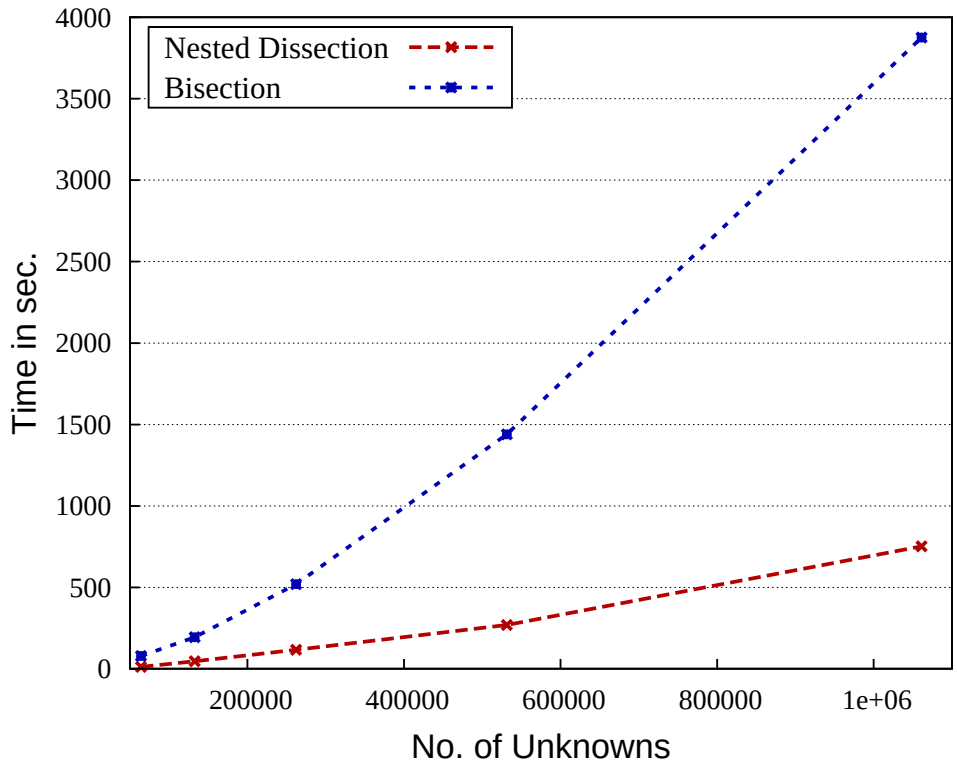
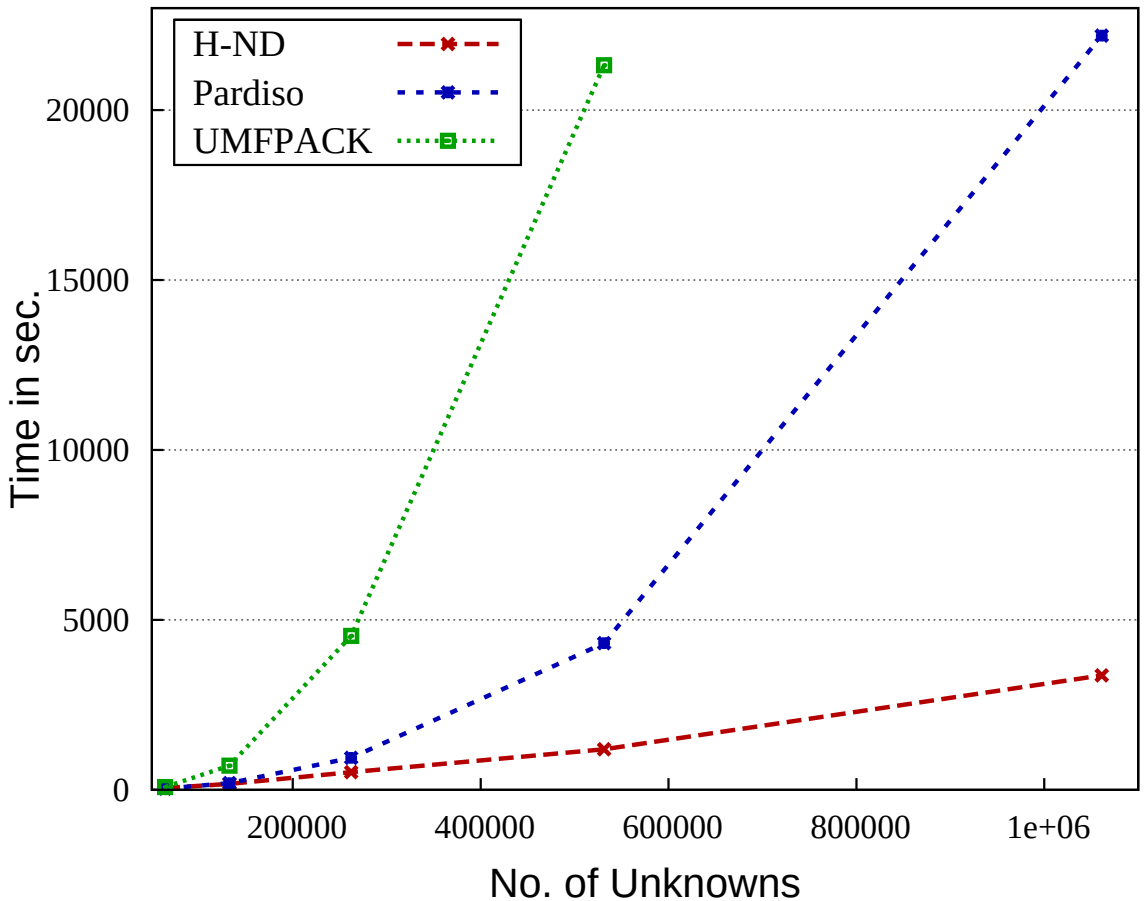


- computation of Γ and clustering $\mathcal{I}$ can be done purely algebraically using graph partitioning for the sparse matrix.

Nested Dissection: Remarks

- $\mathcal{H}$-matrices with nested dissection faster than optimised direct solvers:



Overview

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For all algorithms, the achievable speedup is high compared to the implementation costs.
Alternative: $\mathcal{H}$-Lib^{pro} already implements all presented algorithms and much more.

Literature

R. Kriemann,

Parallele Algorithmen für $\mathcal{H}$-Matrizen,

Ph.D. Thesis, University of Kiel, 2005.

R. Kriemann,

Parallel $\mathcal{H}$-Matrix Arithmetics on Shared Memory Systems,

Computing, 74:273–297, 2005.

M. Bebendorf and R. Kriemann,

Fast Parallel Solution of Boundary Integral Equations and Related Problems,

Computing and Visualization in Science, 8(3–4):121–135, 2005.

L. Grasedyck, R. Kriemann and S. Le Borne,

Domain Decomposition Based $\mathcal{H}$-LU Preconditioning,

submitted to "Numerische Mathematik".

L. Grasedyck, R. Kriemann and S. Le Borne,

Parallel Black Box $\mathcal{H}$-LU Preconditioning for Elliptic Boundary Value Problems,

to appear in "Computing and Visualization in Science".